

# On intersecting families of independent sets in trees

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October 27, 2016

## Abstract

One of the fundamental theorems in extremal combinatorics, the Erdős–Ko–Rado theorem considers intersecting families of subsets of a finite set. For  $1 \leq r \leq n/2$ , it proves that the maximum size of an intersecting family of  $r$ -subsets of an  $n$ -element set is bounded by the size of a *star* family, i.e. a family of all  $r$ -subsets containing a fixed element, called the center of the star. In this note, we consider a recent graph-theoretic generalization of the theorem that looks at intersecting families of independent sets in graphs. In particular, we focus on a special class of trees and prove results concerning centers of maximum-sized star families in these trees.

## 1 Introduction

Let  $[n] = \{1, \dots, n\}$ . Let  $2^{[n]}$  and  $\binom{[n]}{r}$  denote the family of all subsets and  $r$ -subsets of  $[n]$  respectively. A family of subsets  $\mathcal{F} \subseteq 2^{[n]}$  is *intersecting* if  $F \cap G \neq \emptyset$  for  $F, G \in \mathcal{F}$ . For any  $\mathcal{F} \subseteq 2^{[n]}$  and  $x \in [n]$ , let  $\mathcal{F}_x$  be all sets in  $\mathcal{F}$  that contain  $x$ . A classical result of Erdős, Ko and Rado [9] states that if  $\mathcal{F} \subseteq \binom{[n]}{r}$  is intersecting for  $r \leq n/2$ , then  $|\mathcal{F}| \leq \binom{n-1}{r-1}$ . Moreover, if  $r < n/2$ , equality holds if and only if  $\mathcal{F} = \binom{[n]}{r}_x$  for some  $x \in [n]$ . This was shown as part of a stronger result by Hilton and Milner [15] which characterized the structure of the “second-best” intersecting families.

There have been multiple proofs of the Erdős–Ko–Rado theorem. The original proof, by Erdős, Ko and Rado devised the now-central *shifting* technique and used it in conjunction with an induction argument to prove the theorem. Daykin [7] demonstrated that the theorem is implied by the Kruskal–Katona theorem. Katona [20] provided possibly the simplest and most elegant proof, a double counting argument using the method of cyclic permutations. More recently, Frankl–Füredi [11] gave another short proof that relied on a result of Katona on shadows of intersecting families, while we [19] provided an injective

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proof using the aforementioned shifting technique. There have also been algebraic proofs, one using Delsarte's linear programming bound (see [13] and [14] for details), and another using the method of linearly independent polynomials due to Füredi et al. [12].

The Erdős–Ko–Rado theorem is one of the fundamental theorems in extremal combinatorics, and has been generalized in many directions. A very fine survey of the the avenues of research, pursued as extensions of the Erdős–Ko–Rado theorem, in the 1960's, 70's and 80's, is presented by Deza and Frankl [8]. In this note, we focus on a relatively recent graph-theoretic extension of the theorem.

## 1.1 Erdős–Ko–Rado graphs

For a graph  $G$  and integer  $t \leq \alpha(G)$ , where  $\alpha(G)$  is the size of the maximum independent set in  $G$ , we define  $\mathcal{I}^t(G)$  to be the family of all independent sets of  $G$  having size  $t$ . For any family  $\mathcal{F}$  of subsets of  $V(G)$  we denote by  $\mathcal{F}_x$  those sets of  $\mathcal{F}$  that contain the vertex  $x$ . We call  $\mathcal{I}_x^t$  the *star centered on  $x$* , and call  $x$  the *star center*. Call a graph  $G$   $t$ -EKR if, for any  $\mathcal{F} \subseteq \mathcal{I}^t(G)$ ,  $|\mathcal{F}| \leq \max_{x \in V(G)} |\mathcal{I}_x^t(G)|$ .

Earlier results by Berge [2], Deza and Frankl [8], and Bollobas and Leader [3], while not explicitly stated in graph-theoretic terms, hint in this direction. The formulation was initially motivated by a conjecture of Holroyd, who asked if the cycle graph on  $n$  vertices is  $t$ -EKR for every  $t \geq 1$ . Holroyd's conjecture was later proved by Talbot [21]. The formulation also has connections with a fundamental conjecture of Chvátal [6] on intersecting subfamilies of hereditary (closed under subsets) set systems.

Holroyd and Talbot [17] made the following interesting conjecture about the EKR property of graphs. Let  $\mu(G)$  be the size of the smallest *maximal* independent set in  $G$ .

**Conjecture 1.1.** *For a graph  $G$ , let  $1 \leq t \leq \mu(G)/2$ . Then  $G$  is  $t$ -EKR.*

Conjecture 1.1 appears hard to prove in general, but has been verified for certain graph classes. In the paper that introduced this graph-theoretic formulation of the EKR problem, Holroyd, Spencer and Talbot [16] proved the conjecture for a disjoint union of complete graphs, paths and cycles containing at least one isolated vertex. Borg and Holroyd [5] later proved the conjecture for a certain class of interval graphs containing an isolated vertex. In [18], we extended this result and verified the conjecture for all chordal graphs containing an isolated vertex.

One of the reasons why verifying the conjecture for graph classes without isolated vertices is harder is that the intermediate problem of finding the center of the largest star is difficult. (It is easy to see that in a graph containing an isolated vertex, this center is at the isolated vertex.) In this note, we consider this problem for trees.

In [18], we proved that for any tree  $T$  and  $t \leq 4$ ,  $\mathcal{I}_x^t(T)$  is maximum when  $x$  is a leaf. We also conjectured that this is true for every  $t \geq 1$ . However, Baber [1], Borg [4], and Feghali–Johnson–Thomas [10] have separately shown

that this conjecture is not true. This makes it interesting to consider for which trees the conjecture is true.

The authors of [10] consider a special class of trees called spiders, trees obtained from the star graph  $K_{1,n}$  (for some  $n \geq 1$ ) by multiple subdivisions of edges. They prove that two families of spiders, namely the family of all spiders obtained by subdividing each edge of the star graph exactly once, and also the family of all spiders containing one leaf vertex adjacent to the root vertex, satisfy Conjecture 1.1. Note that in both of these subfamilies of spiders, it is easy to find a vertex that is the center of a largest  $t$ -star (for any  $t \geq 1$ ).

In this note, we focus on the problem of determining the centers of the largest stars in all spiders. We first introduce some notation to describe spider graphs.

## 1.2 Spiders

Given a sequence of positive integers  $L = (l_1, \dots, l_k)$  we define the *spider*  $S = S(L)$  to be the tree defined as follows. The *head* of  $S$  is the vertex  $v_0$  and, for  $1 \leq i \leq k$ , the *leg*  $S_i$  is the path  $v_0, v_{i,1}, \dots, v_{i,l_i}$ . We say that  $L$  is in *spider order* if the following conditions hold:

1. if  $l_i$  and  $l_j$  are both odd and  $l_i < l_j$  then  $i < j$ ,
2. if  $l_i$  and  $l_j$  are both even and  $l_i < l_j$  then  $i > j$ , and
3. if  $l_i$  is odd and  $l_j$  is even then  $i < j$ .

To simplify the notation somewhat, we will write  $\mathcal{I}_i^t(G)$  in place of the more cumbersome  $\mathcal{I}_{v_i}^t(G)$ .

## 2 Star Centers

**Theorem 2.1.** *Let  $S = S(L)$  be a spider with  $L = (l_1, \dots, l_k)$  and suppose that  $t \leq \alpha(G)$ . Then for each  $1 \leq i \leq k$  and  $1 \leq j < l_i$  we have  $|\mathcal{I}_{i,j}^t(G)| \leq |\mathcal{I}_{i,l_i}^t(G)|$ .*

**Proof.** We define an injection  $f : \mathcal{I}_{i,j}^t(G) \rightarrow \mathcal{I}_{i,l_i}^t(G)$ .

Let  $A \in \mathcal{I}_{i,j}^t(G)$  and consider the path  $P = v_{i,j}, \dots, v_{i,l_i}$ . For  $0 \leq h \leq (l_i - j)$  we define  $B$  by placing  $v_{i,l_i-h} \in B$  if and only if  $v_{i,j+h} \in A$  —  $B$  is the *flip* of  $A$  on  $P$ , denoted  $\text{flip}_P(A)$ . Let  $T = A - P$ ; then set  $f(A) = B \cup T$ .

Clearly,  $f(A)$  is independent, contains  $v_{i,l_i}$ , and has size  $t$ . Also, if  $f(A') = f(A)$ , then  $A' = A$ .  $\square$

**Theorem 2.2.** *Let  $S = S(L)$  be a spider with  $L = (l_1, \dots, l_k)$  and suppose that  $t \leq \alpha(G)$ . Then for every  $1 \leq i \leq k$  we have  $|\mathcal{I}_0^t(G)| \leq |\mathcal{I}_{i,l_i}^t(G)|$ .*

**Proof.** For fixed  $i$  we define an injection  $f : \mathcal{I}_0^t(G) \rightarrow \mathcal{I}_{i,l_i}^t(G)$ .

First we define  $f$  to be the identity on  $\mathcal{I}_0^t(G) \cap \mathcal{I}_{i,l_i}^t(G)$ .

Second, let  $A \in \mathcal{I}_0^t(G)$  and consider the leg  $S_i = v_0, v_{i,1}, \dots, v_{i,l_i}$ . Write  $v_{i,0} = v_0$  and, for  $0 \leq h \leq (l_i)$  we define  $B$  by placing  $v_{i,l_i-h} \in B$  if and only if  $v_{i,h} \in A$  —  $B$  is the *flip* of  $A$  on  $S_i$ , denoted  $\text{flip}_{S_i}(A)$ . Let  $T = A - S_i$ ; then set  $f(A) = B \cup T$ .

Clearly,  $f(A)$  is independent, contains  $v_{i,l_i}$ , and has size  $t$ . Also, if  $f(A') = f(A)$ , then  $A' = A$ .  $\square$

Together, Theorems 2.1 and 2.2 verify that for the family of spiders, maximum stars are centered at leaves. In what follows, we not only find the best leaf of a spider but give a complete ordering of its leaves according to star size.

**Theorem 2.3.** *Let  $S = S(L)$  be a spider with  $L = (l_1, \dots, l_k)$  in spider order and suppose that  $t \leq \alpha(G)$ . Then for each  $1 \leq i < j \leq k$  we have  $|\mathcal{I}_{i,l_i}^t(G)| \geq |\mathcal{I}_{j,l_j}^t(G)|$ .*

**Proof.** We define an injection  $f : \mathcal{I}_{j,l_j}^t(G) \rightarrow \mathcal{I}_{i,l_i}^t(G)$ . There will be three cases to consider, depending on the parities of  $l_i$  and  $l_j$ . First, we develop some terminology.

For a set  $A \in \mathcal{I}^t(G)$  we can define its ladder as follows. A pair of vertices  $\{v_{i,h}, v_{j,h}\}$  ( $1 \leq h \leq \min(l_i, l_j)$ ) is called a *rung*, which we say is *odd* or *even* according to the parity of  $h$ . A rung is *full* if both its vertices are in  $S$ . The *ladder*  $\mathcal{L}$  of  $A$  is the set of either all even or all odd rungs, depending on whether  $v_0 \in A$  or not, respectively.  $\mathcal{L}$  is *full* if all its rungs are full. If  $\mathcal{L}$  is not full then there is a first (i.e. closest to  $v_0$ ) non-full rung  $R$ . The partial ladder  $\mathcal{L}'$  is the set of all (necessarily full) rungs above  $R$ . Let  $T$  denote those vertices of  $A - \{v_0\}$  not on  $S_i \cup S_j$ .

First we define  $f$  to be the identity on  $\mathcal{I}_{i,l_i}^t(G) \cap \mathcal{I}_{j,l_j}^t(G)$ .

Next we define the function  $f$  on the remaining sets  $A \in \mathcal{I}_{j,l_j}^t(G)$  having partial ladders. Define the path  $P$  from  $v_{j,l_j}$ , up its leg to  $R$ , across  $R$ , and down the other leg to  $v_{i,l_i}$ ; i.e.  $P = v_{j,l_j}, \dots, v_{j,h}, v_{i,h}, \dots, v_{i,l_i}$ , where  $R = (v_{i,h}, v_{j,h})$ . Now slide  $A$  along  $P$  until it contains  $v_{i,l_i}$  — the result we call  $\text{slide}_P(A)$ . Then set  $f(A) = \mathcal{L}' \cup \text{slide}_P(A) \cup T$ .

Of course  $|f(A)| = |A|$ ,  $v_{i,l_i} \in f(A)$ , and  $f(A)$  is independent because  $R$  was not full. Moreover,  $\mathcal{L}'(f(A)) = \mathcal{L}'(A)$ , and so the inverse of  $f$  on  $f(A)$  is uniquely determined.

Note that in these first two cases  $f$  preserves both inclusion and exclusion of  $v_0$ . This means that  $T$  cannot affect the independence of  $f(A)$ .

Finally we define  $f$  on the remaining sets  $A$  having full ladders. If  $l_j$  and  $l_i$  are both even then the full ladder implies that  $v_0 \in A$  because  $v_{j,l_j} \in A$ . If  $l_j$  and  $l_i$  are both odd then, since  $v_{i,l_i} \notin A$ , the full ladder again implies that  $v_0 \in A$ . If  $l_j$  is even and  $l_i$  is odd then  $l_j < l_i$  means that  $v_0 \in A$  because  $v_{j,l_j} \in A$ , while  $l_i < l_j$  means that  $v_0 \in A$  because  $v_{i,l_i} \notin A$ . So in all these remaining cases we have  $v_0 \in A$ .

When  $l_j < l_i$  we let  $P$  be the  $v_{j,l_j}v_{i,l_j-1}$ -path in  $S$  (i.e.  $P = v_{j,l_j}, \dots, v_{j,1}, v_0, v_{i,1}, \dots, v_{i,l_j-1}$ ). When  $l_j > l_i$  we let  $P$  be the  $v_{j,l_i-1}v_{i,l_i}$ -path in  $S$  (i.e.  $P = v_{j,l_i-1}, \dots, v_{j,1}, v_0, v_{i,1}, \dots, v_{i,l_i}$ ). In both cases we let  $Q$  be the  $v_{j,l_j}v_{i,l_i}$ -path in  $S$ , minus  $P$ . We shift  $A$  along  $P$  just one step toward  $v_{i,l_i}$  — call the result  $\text{shift}_P(A)$  — and flip  $A$  on  $Q$  (that is, if  $Q = (q_0, \dots, q_k)$  then replace each  $q_h$  in  $A$  by  $q_{k-h}$ ) — call the result  $\text{flip}_Q(A)$ . Now define  $f(A) = \text{shift}_P(A) \cup \text{flip}_Q(A) \cup T$ . Of course  $|f(A)| = |A|$ ,  $v_{i,l_i} \in f(A)$  (because of the flip if  $l_j < l_i$  or the shift if  $l_j > l_i$ ), and  $A$  is independent (because of the flip if  $l_j < l_i$  or the shift if  $l_j > l_i$ ). Moreover,  $f(A)$  has a full ladder, and so the inverse of  $f$  on  $f(A)$  is uniquely determined.

Notice that, because of the shift,  $v_0 \notin f(A)$ , and so  $T$  cannot affect the independence of  $f(A)$ . Thus the injection is complete.  $\square$

### 3 Open questions

Determining whether or not spider graphs satisfy Conjecture 1.1 remains open. The compression/induction technique that has been used to prove Conjecture 1.1 for other graph classes appears difficult to use in this case. The nature of Theorem 2.3 implies that the center of the largest star may “jump” when we consider subtrees of the spider.

In general, determining the centers of the largest stars in trees remains an open problem.

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